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ON THE VARIETY OF RATIONAL SPACE CURVES

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ABSTRACT

We consider the locus of smooth rational curves of given degree in a given projective space, which are incident to a generic collection of linear spaces. When this locus is finite (resp. 1-dimensional) we give a recursive procedure to compute its degree (resp. geometric genus). The method is based on the elementary geometry of ruled surfaces.

Introduction

In recent years it has become customary to approach certain types of problems of enumerative geometry via the rather substantial machinery of quantum cohomology — chiefly compactified moduli spaces of maps and related objects (cf. [3]) — leading some to believe, perhaps, that such machinery may be indispensable.

In this paper we study enumeratively the variety $V_{d,n}$ parametrising irreducible nonsingular rational curves of degree d in $\mathbb{P}^n, n \geq 3$. We shall give a recursive procedure which computes two sets of numbers associated to $V_{d,n}$:

- The Schubert degrees $N_{d,n}(a_1, \ldots, a_k)$, i.e. the degree of the locus of members of $V_{d,n}$ meeting a generic collection $A_1, \ldots, A_k$ of linear subspaces of respective codimensions $a_1, \ldots, a_k$ in $\mathbb{P}^n$, whenever $\sum(a_i - 1) = (n+1)d + (n-3) = \dim V_{d,n}$, i.e. whenever the locus in question is finite;
- the linear genera $g_{d,n}(a_1, \ldots, a_k)$, i.e. the (geometric) genus of the analogously-defined locus whenever it is 1-dimensional, i.e. whenever $\sum(a_i - 1) = \dim V_{d,n} - 1$ (in which case it is in fact smooth, except for a

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certain number of ordinary nodes if some $a_i = 2$ (see Remark at the end of Section 2).

Analogous questions for $n = 2$ were considered in [6], [7] (degree questions for $n = 2$ were also considered by Caporaso and Harris [1]). As there, the method is completely elementary, involving some geometry on ruled surfaces. No quantum-cohomological methods are used; indeed our method may be viewed as an alternative to those (compare [3] and references therein). The new issues arising vis-a-vis [6], [7] are principally two:

(i) the 'distinguished sections' on our ruled surface X are no longer necessarily disjoint: this necessitates setting up, in addition to the ever-present degree recursion, a further recursion involving the effective length of the condition-vector $(a_1, \ldots a_k)$;

(ii) the target projective space being of dimension $> 2 = \dim X$, the 'Riemann–Hurwitz' or ramification formula is not immediately available: to remedy this we 'thicken' our ruled surface to a (noncompact, so as to avoid bad curves) ruled n-fold.

While our degree formulae are in principle derivable from the general methods of quantum cohomology, the genus formulae are apparently new for $n \geq 3$; for $n = 2$ analogous formulae were given, using (resp. not using) quantum cohomology, by Pandharipande [5] (resp. by ourselves [7]). Interestingly, our recursive procedure is meaningful even for lines ($d = 1$), yielding a procedure for computing intersection numbers of 'primary' Schubert cycles (consisting of lines meeting a given linear space) and for the linear genus of a grassmannian of lines in $\mathbb{P}^n$.

1. Degrees

In what follows we fix $n \geq 3$ and denote by $\bar{V}_d$ or $\bar{V}_{d,n}$ the closure in the Chow variety of the locus of irreducible nonsingular rational curves of degree d in $\mathbb{P}^n$, with the scheme structure as closure, i.e. the reduced structure. Let $A_1, \ldots, A_k$ be a generic collection of linear subspaces of respective codimensions $a_1, \ldots, a_k \geq 1$ in $\mathbb{P}^n$. We denote by

$$B_d = B_d(a.) = B_d(A.)$$

the normalization of the locus (with reduced structure)

$$\{(C, P_1, \ldots, P_k) \colon C \in \bar{V}_d, P_i \in C \cap A_i, i = 1, \ldots, k\}$$

when all $a_i > 1$; this is also the normalization of its projection to $\bar{V}_d$, i.e. the locus of degree-d rational curves (and their specializations) meeting $A_1, \ldots, A_k$;

however, it will be convenient to allow some A_i to be hyperplanes. Of course, if some $a_i > n$ then $N_d(a.) = 0$. Let us call the number of i such that $a_i > 1$ the **length** of the condition vector $(a.)$. We have

$$\dim B = (n+1)d + (n-3) - \sum(a_i - 1).$$

When this is 0, we set

$$N_d(a.) = \deg B_d(a.).$$

Of course $N_d(1, a_2, \ldots) = dN_d(a_2, \ldots)$, so it will suffice to compute these when all $a_i > 1$.

The plan is to get at them via suitable 1-dimensional B's. To this end, take a $B = B(A.)$ 1-dimensional and let

$$\pi\colon X \to B$$

be the normalization of the tautological family of rational curves (always with reduced structure), and $f\colon X \to \mathbb{P}^n$ the natural map. We begin with some elementary, but possibly surprising, qualitative remarks, whose import is that, thanks chiefly to the fact that the incidence conditions to $(A.)$ are 'insensitive' to scheme structure, possible pathologies of the Hilbert scheme 'don't matter' and the curves in B are well behaved.

LEMMA 1.0: *Each fibre F of π is either*

(i) *a $\mathbb{P}^1$ on which f is an embedding; or*

(ii) *a pair of $\mathbb{P}^1$'s meeting transversely once, on which f is an embedding; or*

(iii) *if $n = 3$, a $\mathbb{P}^1$ on which f is a degree-1 immersion such that $f(\mathbb{P}^1)$ has a unique singular point which is an ordinary node.*

Proof: Firstly, it is easy to see that, in the family of parametrised rational curves $g\colon \mathbb{P}^1 \to \mathbb{P}^n$, the locus of those where g is not an embedding $(n > 3)$, or has mapping degree > 1 or is not an immersion or where $g(\mathbb{P}^1)$ has worse singularities than one ordinary node $(n = 3)$, has codimension ≥ 2, hence there will be no such curves $g(\mathbb{P}^1)$ satisfying the incidence conditions to $(A.)$. Next, if F has j components $F_1, \ldots, F_j$, $F_i \simeq \mathbb{P}^1$, with $f(F_i)$ of degree d_i, $\sum d_i \leq d$, then, because there are $n-2$ conditions for a pair of rational curves in $\mathbb{P}^n$ to meet, the curve $f(F)$ must a priori belong to a family of dimension at most

$$\sum_{i=1}^{k}[(n+1)d_i + (n-3)] - (k-1)(n-2) = (n+1)d + (n-3) - (k-1),$$

and since $f(F)$ must also satisfy the incidence conditions to $(A.)$, it follows that $k = 2$ and $d_1 + d_2 = d$. By similar considerations, $f(F_1), f(F_2)$ are smooth and meet transversely once. ∎

COROLLARY 1.1: *(i) $\bar{V}_{d,n}$ is smooth along the image $\bar{B}$ of B, and $\bar{B}$ is smooth except for ordinary nodes corresponding to curves meeting some A_i of codimension 2 transversely twice;*

(ii) X is smooth.

Proof: (i) It follows from the Lemma that *Hilb* itself, a foriori $\bar{V}_{d,n}$, is smooth along $\bar{B}$ except possibly for $n = 3$ at points corresponding to an immersed nonembedded curve f_1: $\mathbb{P}^1 \to \mathbb{P}^3$ where the corresponding subscheme C of $\mathbb{P}^3$ has an embedded point. Now a neighborhood of C in $\bar{V}_{d,n}$ is bijectively parametrised by a space H parametrising deformations of the parametrised curve, which has tangent space $H^0(N_f)$ and obstruction space $H^1(N_f) = 0$ (as N_f is globally generated); thus H is smooth locally. Moreover, a nonzero element of $H^0(N_f)$ induces a deformation of the cycle $f(\mathbb{P}^1)$ which is locally nontrivial at a generic point, hence nonzero. Thus the natural map $H \to \bar{V}_{d,n}$ is bijective and unramified, hence an isomorphism locally, so $\bar{V}_{d,n}$ is smooth at C. Applying similar considerations to the deformations preserving incidence to $(A.)$, we see that $\bar{B}$ is smooth too except for the singularities noted.

(ii) This is a purely local assertion at singular points of reducible fibres which follows either by a little deformation theory for reducible curves as in [8] or alternatively by generically projecting to $\mathbb{P}^2$ and using the corresponding fact there (cf. [2]). ∎

Motivated by this result we introduce as in [7] two additional set of numbers: $N_d(2\times, a_2, \ldots, a_k)$ (resp. $N_d(2 \to, a_2, \ldots, a_k)$) denotes the number of rational curves of degree d incident to $A_2, \ldots, A_k$ of respective codimension a_i and meeting A_1 of codimension 2 twice (resp. meeting A_1 once and tangent along A_1 to a fixed hyperplane containing A_1.) As in [7] we have the basic identity

$$(0) \qquad N_d(2, 2, a_2, \ldots, a_k) = N_d(2 \to, a_2, \ldots, a_k) + 2N_d(2\times, a_2, \ldots, a_k).$$

Now let $\mathcal{F}$ be the set of components of reducible fibres of π. Note X comes equipped with a set of distinguished sections $s_i = s_{A_i}, i = 1, \ldots, k$ and note that

$$s_i.s_j = N_d(\ldots, a_i + a_j, \ldots, \hat{a}_j, \ldots), \quad i \neq j.$$

Also, let $R_i = R_{A_i}$ be the sum of all fibre components not meeting s_{A_i} and $\mathcal{F}_A \subset \mathcal{F}$ the set of such components. Then R_{A_i} may by blown down, giving rise to a geometrically ruled surface:

$$b_i \colon X \to X_i = X_{A_i} = \mathbb{P}(E_{A_i}), \tag{1}$$

with sections $\bar{s}_j = b_i(s_j)$, and note that if, say, $i = 1$, then

$$\bar{s}_2.\bar{s}_1 = s_2.s_1.$$

Now set

$$m_i = m_i(a_1, \ldots, a_k) = -s_i^2, \quad i = 1, \ldots, k.$$

As in [7], we see that

$$m_1(2, a_2, \ldots) = N_d(2 \to, a_2, \ldots).$$

Note that, for any $a.$, $m_1 = -\bar{s}_1^2$ too. But clearly, on a geometric ruled surface the difference of any two sections is a sum of fibres, hence it has self-intersection $= 0$, hence

$$\begin{aligned} 0 &= (\bar{s}_1 - \bar{s}_2)^2 \\ &= s_1^2 + s_2^2 + s_2 \cdot R_1 - 2s_2 \cdot s_1, \\ &= -m_1 - m_2 + s_2 \cdot R_1 - 2s_2 \cdot s_1, \end{aligned}$$

i.e.

$$m_1 + m_2 = s_2 \cdot R_1 - 2N_d(a_1 + a_2, a_3, \ldots). \tag{2}$$

One consequence of this, already noted and used in [1], is the

2-Section Lemma 1.2: *If $a_1 = a_2$, then*

$$s_1^2 = s_2^2 = \frac{-1}{2} s_1 \cdot R_2 + N_d(2a_1, a_3, \ldots). \tag{3}$$

Indeed, if $a_1 = a_2$ then clearly by monodromy $m_1 = m_2$ so (3) follows from (2).

For general codimensions we have the

3-Section Lemma 1.3: *For any 3 distinct distinguished sections s_1, s_2, s_3 we have*

$$s_1^2 = \frac{-1}{2}(s_1 \cdot R_2 + s_1 \cdot R_3 - s_2 \cdot R_3) + s_1.s_2 + s_1.s_3 - s_2.s_3. \tag{4}$$

This follows immediately from (2) by a suitable linear combination.

Note that by an obvious dimension count the number k of distinguished sections on X is ≥ 3, so that Lemma 1.2 is applicable, except in the single case $d =$

$1, (a.) = (n, n-1)$, where one is considering the set of lines through a point meeting a fixed line, so clearly $s_1^2 = -1, s_2^2 = 1$. Also, from a recursive standpoint, numbers such as $s_1 \cdot R_2$, having to do with reducible curves, are easily computable in terms of $N_{d'}, d' < d$, hence they may be considered known. Indeed,

$$s_1 \cdot R_2 = \sum N_{d_1}(A^1_., A_1, \mathbb{P}^{s_1}) N_{d_2}(A^2_., A_2, \mathbb{P}^{s_2}), \tag{5}$$

the summations being over all $d_1 + d_2 = d, s_1 + s_2 = n$ and all decompositions $A_. = (A_1, A_2) \coprod (A^1_.) \coprod (A^2_.)$ (as unordered sequences or partitions). Each term corresponds to a pair of families of curves each of degree d_i meeting $(A_1, A^i_.)$ and filling up a locus of codimension s_i and degree $N_{d_i}(A^i_., A_i, \mathbb{P}^{s_i})$ in $\mathbb{P}^n$, $i = 1, 2$; the two loci meet in a finite set whose cardinality is given by Bezout's theorem and whose members correspond with $s_1 \cap R_2$.

Thus at least when $a_1 + a_2, a_1 + a_3, a_2 + a_3$ are all $> n$ (which is automatic if $n = 3$ but not otherwise), the 3-section lemma computes m_1, m_2, m_3 in terms of lower-degree data; but even if this condition is not satisfied (and, say, $a_1, a_2, a_3 > 1$), the lemma still computes m_1, say, in terms of data of lower degree *or lower length*; we shall use this observation below in constructing a 'length recursion'.

Now let $L = f^*\mathcal{O}(1)$ and F_0 be a general fibre of π. Then we have

$$L = ds_1 - \sum_{F \in \mathcal{F}_{A_1}} \deg(F) F + x F_0 \tag{6}$$

for some $x \in \mathbb{Q}$: indeed (6) holds simply because both sides have the same value on all fibre components. To determine x, evaluate on s_1, noting that, by definition,

$$L \cdot s_1 = N_d(a_1 + 1, a_2, \ldots).$$

Thus we have

$$x = N_d(a_1 + 1, a_2, \ldots) + dm_1.$$

Now let us square (6), noting that, by definition, $L^2 = N_d(2, a_1, \ldots)$. Thus

$$N_d(2, a_1, \ldots) = -d^2 m_1 + 2d^2 m_1 + 2dN_d(a_1 + 1, a_2, \ldots) - \sum_{F \in \mathcal{F}_{A_1}} (\deg F)^2,$$

i.e., denoting $\mathcal{F}_{A_1}$ by $\mathcal{F}_1(a.)$, we have

$$N(2, a_1, \ldots) = 2dN_d(a_1 + 1, a_2, \ldots) + d^2 m_1(a.) - \sum_{F \in \mathcal{F}_1(a.)} (\deg F)^2. \tag{7}$$

As above, the sum $\sum_{F \in \mathcal{F}_A} (\deg F)^2$ is easily evaluated in terms of $N_{d'}, d' < d$ and may be considered known. In particular, when $a_1 = n$, i.e. A_1 is a point, and moreover $a_2 + a_3 > n$, we have $N_d(a_1+1, a_2, \ldots) = 0$ and $m_1(a.)$, via the 3-section lemma, is computable from lower-degree data, so (7) yields a recursive formula for all the $N_d(2, n, a_2, a_3, \ldots), a_2 + a_3 > n$, namely

$$N_d(2, n, a_2, \ldots) = d^2 m_1(n, a_2, \ldots) - \sum_{F \in \mathcal{F}_1(a.)} (\deg F)^2. \tag{8}$$

Now take the dot product of (6) with s_2, obtaining

$$\begin{aligned} N_d(a_1, a_2+1, \ldots) =& dN_d(a_1+a_2, \ldots) \\ & - \sum_{F \in (\mathcal{F}_1 - \mathcal{F}_2)(a.)} (\deg F) + N_d(a_1+1, a_2, \ldots) + dm_1(a.). \end{aligned} \tag{9}$$

Now to determine $N_d(a.)$ in general we proceed by recursion on k (as well as d), as follows. We may assume all $a_i > 1$. For the smallest possible k (given $d \geq 3$), clearly we may assume by reordering that $a_1 = a_2 = a_3 = n$ so, from (7) (read backwards), we compute $N_d(a.)$ from $N_d(2, n-1, n, n, \ldots) = N_d(2, n, n, n-1, \ldots)$ (and lower-degree data). But $N_d(2, n, n, n-1, \ldots)$ has already been computed above; this takes care of the case of smallest k. In the general case we use recursion on k. Using (9), we compute $N_d(a.)$ from $N_d(a_1+1, a_2-1, \ldots)$ and terms of lower degree or length. Applying (9) repeatedly we compute $N_d(a.)$ in terms of lower-degree and lower-length data plus $N_d(a_1+a_2-1, 1, a_3, \ldots) = dN_d(a_1+a_2-1, a_3, \ldots)$, itself a lower-length term. This computes $N_d(a.)$ in general. Note that the lowest-degree, lowest-length term, i.e. the initial case of the recursion, is $N_1(n, n) = 1$, i.e. the unique line through 2 points.

We now illustrate the use of the above formulae for $n = 3$. For brevity we shall assume known the 'line' numbers $N_1(2^2, 3) = 1, N_1(2^4) = 2$, as well as the conic numbers

$$N_2(2^4, 3^2) = 4, \quad N_2(2^6, 3) = 18, \quad N_2(2^8) = 92,$$

which are similar to but simpler than cubic numbers, and proceed to compute the cubic numbers $N_3(2^{10}, 3)$ and $N_3(2^{12})$ (other cubic numbers are again similar but simpler).

First from (8) we have

$$N_3(2^{10}, 3) = N_3(2, 3, 2^9) = 9m_1(3, 2^9) - \sum_{F \in \mathcal{F}_1(3, 2^9)} \deg(F)^2. \tag{$*$}$$

To compute the m_1 it is best to use the idea of Lemma 1.3 rather than the Lemma itself. First (2) yields

$$m_1 + m_2 = s_1.R_2.$$

The latter counts certain intersecting pairs (line, conic) or (conic, line) whose first element contains the point A_1 and the second meets the line A_2 and which between them satisfy the incidence conditions to the other A_i; moreover, the contribution from each type is a sum of two terms depending on which component is determined up to finite choice by incidence to the A_i alone (then the other component is determined up to finite choice by incidence to the remaining A_i and to the previous component). It is easy to compute that the two contributions are, respectively,

$$(\binom{8}{2} + 8.2)N_2(2^8) \quad \text{and} \quad (\binom{8}{6}.4 + \binom{8}{3}.2)N_2(3,2^6).$$

Consequently

$$m_1 + m_2 = 8080.$$

Similarly,

$$2m_2 = m_2 + m_3 = s_2.R_3 = 7200, \quad m_2 = 3600.$$

Hence

$$m_1 = 4480.$$

The other term in $(*)$ can be computed similarly and equals 30456, hence

$$N_3(2^{10},3) = 9864.$$

Next, to compute $N_3(2^{12})$ we use (7), which yields

$$(**) \qquad N_3(2^{12}) = 6N_3(3,2^{10}) + 9m_1(2^{11}) - \sum_{F\in\mathcal{F}_1(2^{11})} \deg(F)^2.$$

Again it is straightforward to compute the second and third terms and we get

$$m_1 = 28704, \quad \sum \deg(F)^2 = 237360.$$

Consequently, we deduce the classical count for the number of twisted cubics meeting 12 general lines:

$$N_3(2^{12}) = 80160.$$

2. Genera

In this section we fix a sequence $(a.)$ giving rise to a smooth (maybe disconnected) curve $B = B(a.)$ and give a formula for the latter's geometric genus, i.e. for $\deg(K_B)$. The idea is to consider a 'thickening'

$$B \to B^+ = B(a^+_.) = B(A^+_.), \quad \dim B^+ = n-1,$$

where $A. = (A_1, \ldots, A_k)$,

$$\begin{aligned} A^+_\cdot &= (A_1^+, \ldots, A_\ell^+), \quad \ell \le k, \\ A_i &\subseteq A_i^+ \subset \mathbb{P}^n, \\ a_i^+ &= \operatorname{codim} A_i^+ \ge 2. \end{aligned}$$

Consider the diagram (over a neighborhood of the image of B)

$$\begin{array}{ccccc} X & \longrightarrow & X^+ & \xrightarrow{f^+} & \mathbb{P}^n \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & B^+ & & \end{array}$$

with f^+ generically finite. Let ρ be the ramification divisor of f^+. Then, as in [2], it is easy to see that

$$\rho|_X = \sum (a_i^+ - 1)s_i.$$

On the other hand, by Riemann–Hurwitz,

$$-(n+1)L + \rho = K_{X^+}|_X = K_{X/B} + \pi^*(K_{B^+}|_B). \tag{10}$$

Now recall the blowing down map $X \to X_1 = X_{A_1} = \mathbb{P}(E_{A_1})$ (1). It is easy to see that

$$K_{X_1/B} = -2s_1 - m_1 F_0,$$

hence

$$K_{X/B} = -2s_1 - m_1 F_0 + R_1,$$

so that, by (8),

$$\pi^*(K_{B^+}|_B) = -(n+1)L + \sum (a_i^+ - 1)s_i + 2s_1 + m_1 F_0 - R.$$

Evaluating on s_1, we conclude that

$$\deg(K_{B^+}|_B) = -(n+1)N_d(a_1 + 1, a_2, \ldots) - a_1^+ m_1. \tag{11}$$

By the adjunction formula,

$$\deg K_B = \deg(K_{B^+}|_B) + \deg N_{B/B^+}. \tag{12}$$

On the other hand, if we set

$$\begin{aligned} B_i &= B(a_1, \ldots, a_i^+, \ldots, a_k), \quad i = 1, \ldots, \ell \\ &= B(a_1, \ldots, \hat{a}_i, \ldots), \quad i > \ell \end{aligned}$$

then clearly $B = \bigcap B_i$ and, by the standard computation of tangent spaces $T_b B = \bigcap T_b B_i$ for any $b \in B$, hence

$$\deg N_{B/B^+} = \sum_{i=1}^{k} \deg N_{B/B_i}, \tag{13}$$

so it suffices to evaluate $\deg N_{B/B_i}$.

CASE 1: $i \le \ell$

We then have a Cartesian diagram

$$\begin{array}{ccc} s_{A_i} & \longrightarrow & A_i \\ \cap & & \cap \\ s_{A_i^+} & \longrightarrow & A_i^+ \end{array}$$

from which clearly

$$\pi^* N_{B/B_i} = N_{s_{A_i}/s_{A_i^+}} = f^*((a_i - a_i^+)\mathcal{O}(1)),$$

hence

$$\deg N_{B/B_i} = (a_i - a_i^+) N(a_1, \ldots, a_i + 1, \ldots). \tag{14}$$

CASE 2: $i > \ell$

We then have a Cartesian diagram

$$\begin{array}{ccccc} & & s_{A_i} & \longrightarrow & A_i \\ & & \cap & & \\ B & \longleftarrow & X & & \cap \\ \downarrow & & \downarrow & & \\ B_i & \longleftarrow & X_i & \longrightarrow & \mathbb{P}^n \end{array}$$

from which, as above,

$$\deg N_{s_{A_i}/X_i} = a_i N_d(a_1, \ldots, a_i + 1, \ldots).$$

On the other hand,

$$\begin{aligned} \deg N_{s_{A_i}/X_i} &= \deg N_{s_{A_i}/X} + \deg N_{X/X_i}|s_{A_i} \\ &= -m_i + \deg \pi^* N_{B/B_i}. \end{aligned}$$

Consequently

$$\deg N_{B/B_i} = a_i N_d(a_1, \ldots, a_i + 1, \ldots) + m_i. \tag{15}$$

Putting (11)–(15) together, we have computed $\deg(K_B)$, as claimed.

Remark: As in [7], we also get a formula for the arithmetic genus of $\bar{B}$ if $a_1 = 2$, namely, if $t = \#\{i : a_i = 2\}$,

$$p_a(\bar{B}) = g(B) + tN_d(2\times, a_2, \ldots) = g(B) + \frac{t}{2}(N_d(2, 2, a_2, \ldots) - m_1(2, a_2, \ldots)).$$

Note: For some applications of the results of this paper, and some more details on the qualitative issues related to Lemma 1.0, see the author's eprint math.AG/0002101, entitled *The degree of the divisor of jumping rational curves.*

References

[1] L. Caporaso and J. Harris, *Parameter spaces for curves on surfaces and enumeration of rational curves;* and *Enumerating rational curves: The rational fibration method,* Compositio Mathematica **113** (1998), 155–208; 209–237.

[2] S. Diaz and J. Harris, *Geometry of Severi varieties,* Transactions of the American Mathematical Society **309** (1988), 1–34.

[3] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology,* Proceedings of the Santa Cruz Symposium on Algebraic Geometry, American Mathematical Society, Providence, RI, 1997, pp. 95–96.

[4] J. Harris, *Algebraic Geometry,* Springer, Berlin, 1994.

[5] R. Pandharipande, *The canonical class of $M_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry,* International Mathematics Research Notices No. 4 (1997), 173–186.

[6] Z. Ran, *Bend, break and count,* Israel Journal of Mathematics **111** (1999), 109–124.

[7] Z. Ran, *Bend, break and count II,* Mathematical Proceedings of the Cambridge Philosophical Society, to appear.

[8] E. Sernesi, *On the existence of certain families of curves,* Inventiones Mathematicae **75** (1984), 25–57.